

Geometrical objects in the jet Hamilton geometry of momenta

Mircea Neagu

Abstract

In this paper we introduce on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$ the main geometrical objects used in the jet geometry of time-dependent Hamiltonians. We talk about distinguished (d-) tensors, time-dependent semisprays, nonlinear connections and their mathematical connections.

Mathematics Subject Classification (2010): 53B40, 53C60, 53C07.

Key words and phrases: dual 1-jet space, d-tensors, time-dependent semisprays of momenta, nonlinear connections, adapted bases.

1 Introduction

According to Olver's opinion [6], we recall that the 1-jet spaces and their duals are fundamental ambient mathematical spaces used in the study of classical and quantum field theories in their Lagrangian and Hamiltonian approaches (see also [3]). For that reason, inspired by the Cartan covariant Hamiltonian approach of classical Mechanics, the studies of Miron [4] and Atanasiu ([1], [2]) led to the development of the *Hamilton geometry of cotangent bundles* exposed by Miron, Hrimiuc, Shimada and Sabău in monograph [5]. We underline that, via the Legendre duality of the Hamilton spaces with the Lagrange spaces, the preceding authors have shown in [5] that the theory of Hamilton spaces has the same symmetry as the Lagrange geometry, giving thus a geometrical framework for the Hamiltonian theory of Analytical Mechanics.

In such a physical and geometrical context, suggested by the cotangent bundle framework of the Miron et al., this paper is devoted to exposing the *time-dependent covariant Hamilton geometry on dual 1-jet spaces* (in the sense of d-tensors, time-dependent semisprays of momenta and nonlinear connections), which is a natural jet extension of the Hamilton geometry on the cotangent bundle from [5].

2 The dual 1-jet space

We start our geometrical study considering a smooth real manifold M^n of dimension n , whose local coordinates are $(x^i)_{i=1,n}$. Let us also consider the *dual*

1-jet vector bundle

$$J^{1*}(\mathbb{R}, M) \equiv \mathbb{R} \times T^*M \rightarrow \mathbb{R} \times M,$$

whose local coordinates are denoted by (t, x^i, p_i^1) .

Remark 1 According to the Hamiltonian terminology from Analytical Mechanics, the coordinates p_i^1 are called **momenta**, and the dual 1-jet space $J^{1*}(\mathbb{R}, M)$ is called the **time-dependent phase space of momenta**.

The transformations of coordinates $(t, x^i, p_i^1) \longleftrightarrow (\tilde{t}, \tilde{x}^i, \tilde{p}_i^1)$, induced from $\mathbb{R} \times M$ on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, have the expressions

$$\begin{cases} \tilde{t} = \tilde{t}(t) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{p}_i^1 = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{d\tilde{t}}{dt} p_j^1. \end{cases} \quad (1)$$

where $d\tilde{t}/dt \neq 0$ and $\det(\partial \tilde{x}^i / \partial x^j) \neq 0$. Consequently, in our dual jet geometrical approach, we use a *relativistic* time t .

Comparatively, in Miron's Hamiltonian approach, the authors use the trivial bundle $\mathbb{R} \times T^*M$ over the base cotangent space T^*M , whose coordinates induced by T^*M are (t, x^i, p_i) . Thus, the changes of coordinates on the trivial bundle

$$\mathbb{R} \times T^*M \rightarrow T^*M$$

are given by

$$\begin{cases} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j, \end{cases}$$

pointing out the *absolute* character of the time variable t . A time dependent Hamiltonian function for M is a real valued function H on $\mathbb{R} \times T^*M$. Such Hamiltonians (which are called *rheonomic*, or *non-autonomous*, or *time-dependent*) are important for covariant Hamiltonian approach of time-dependent mechanics. A geometrization of these Hamiltonians on the trivial bundle $\mathbb{R} \times T^*M \rightarrow T^*M$ is investigated by Miron, Atanasiu and their co-workers in the works [1], [2], [4] and [5].

Now, doing a transformation of coordinates (1) on $J^{1*}(\mathbb{R}, M)$, we obtain

Proposition 2 On the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, the elements of the local natural basis of vector fields

$$\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i^1} \right\} \subset \mathcal{X}(J^{1*}(\mathbb{R}, M))$$

transform by the rules

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{d\tilde{t}}{dt} \frac{\partial}{\partial \tilde{t}} + \frac{\partial \tilde{p}_j^1}{\partial t} \frac{\partial}{\partial \tilde{p}_j^1}, \\
\frac{\partial}{\partial x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}_j^1}{\partial x^i} \frac{\partial}{\partial \tilde{p}_j^1}, \\
\frac{\partial}{\partial p_i^1} &= \frac{\partial x^i}{\partial \tilde{x}^j} \frac{d\tilde{t}}{dt} \frac{\partial}{\partial \tilde{p}_j^1}.
\end{aligned} \tag{2}$$

Proposition 3 *The elements of the local natural cobasis of covector fields*

$$\{dt, dx^i, dp_i^1\} \subset \mathcal{X}^*(J^{1*}(\mathbb{R}, M))$$

transform by the rules

$$\begin{aligned}
dt &= \frac{dt}{d\tilde{t}} d\tilde{t}, \\
dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j, \\
dp_i^1 &= \frac{\partial p_i^1}{\partial \tilde{t}} d\tilde{t} + \frac{\partial p_i^1}{\partial \tilde{x}^j} d\tilde{x}^j + \frac{\partial \tilde{x}^j}{\partial x^i} \frac{dt}{d\tilde{t}} d\tilde{p}_j^1.
\end{aligned} \tag{3}$$

3 Time-dependent semisprays of momenta

Following the geometrical ideas developed in book [5], in our study on the geometry of the dual 1-jet bundle $J^{1*}(\mathbb{R}, M)$ a central role is played by *d-tensors*.

Definition 4 *A geometrical object $T = \left(T_{1j(1)(l)\dots}^{1i(k)(1)\dots}(t, x^r, p_r^1)\right)$ on the dual 1-jet vector bundle $J^{1*}(\mathbb{R}, M)$, whose local components, with respect to a transformation of coordinates (1) on $J^{1*}(\mathbb{R}, M)$, transform by the rules*

$$T_{1j(1)(l)\dots}^{1i(k)(1)\dots} = \tilde{T}_{1q(1)(s)\dots}^{1p(r)(1)\dots} \frac{dt}{d\tilde{t}} \frac{\partial x^i}{\partial \tilde{x}^p} \left(\frac{\partial x^k}{\partial \tilde{x}^r} \frac{d\tilde{t}}{dt} \right) \frac{d\tilde{t}}{dt} \frac{\partial \tilde{x}^q}{\partial x^j} \left(\frac{\partial \tilde{x}^s}{\partial x^l} \frac{dt}{d\tilde{t}} \right) \dots,$$

is called a **d-tensor** or a **distinguished tensor field** on $J^{1*}(\mathbb{R}, M)$.

Remark 5 *The placing between parentheses of certain indices of the local components $T_{1j(1)(l)\dots}^{1i(k)(1)\dots}$ is necessary for clearer future contractions. For the moment, we point out only that $\binom{(k)}{(1)}$ or $\binom{(1)}{(l)}$ behaves like a single **double index**.*

Example 6 *If $H : J^{1*}(\mathbb{R}, M) \rightarrow \mathbb{R}$ is a Hamiltonian function depending on the momenta p_i^1 , then the local components*

$$G_{(1)(1)}^{(i)(j)} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i^1 \partial p_j^1}$$

represent a d -tensor field $\mathbb{G} = \left(G_{(1)(1)}^{(i)(j)} \right)$ on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$. This is called the **vertical fundamental metrical d -tensor** produced by the Hamiltonian function of momenta H .

Example 7 Let us consider the d -tensor $\mathbb{C} = \left(\mathbb{C}_{(i)}^{(1)} \right)$, where $\mathbb{C}_{(i)}^{(1)} = p_i^1$. The distinguished tensor \mathbb{C} is called the **Liouville-Hamilton d -tensor field of momenta** on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$.

Example 8 Let $h_{11}(t)$ be a semi-Riemannian metric on the time manifold \mathbb{R} . The geometrical object $\mathbb{L} = \left(L_{(j)11}^{(1)} \right)$, where $L_{(j)11}^{(1)} = h_{11}p_j^1$, is a d -tensor field on $J^{1*}(\mathbb{R}, M)$. This is called the **momentum Liouville-Hamilton d -tensor field associated with the metric $h_{11}(t)$** .

Example 9 Using the preceding metric $h_{11}(t)$, we construct the d -tensor field $\mathbb{J} = \left(J_{(1)1j}^{(i)} \right)$, where $J_{(1)1j}^{(i)} = h_{11}\delta_j^i$. The distinguished tensor \mathbb{J} is called the **d -tensor of h -normalization** on the dual 1-jet vector bundle $J^{1*}(\mathbb{R}, M)$.

It is obvious that any d -tensor on $J^{1*}(\mathbb{R}, M)$ is a tensor field on $J^{1*}(\mathbb{R}, M)$. Conversely, this statement is not true. As examples, we construct two tensors on $J^{1*}(\mathbb{R}, M)$, which are not distinguished tensors on $J^{1*}(\mathbb{R}, M)$.

Definition 10 A global tensor G_1 on $J^{1*}(\mathbb{R}, M)$, locally expressed by

$$G_1 = p_i^1 dx^i \otimes \frac{\partial}{\partial t} - 2G_{1(j)i}^{(1)} dx^i \otimes \frac{\partial}{\partial p_j^1},$$

is called a **temporal semispray** on the dual 1-jet vector bundle $J^{1*}(\mathbb{R}, M)$.

Taking into account that the temporal semispray G_1 is a global tensor on $J^{1*}(\mathbb{R}, M)$, by a direct calculation, we obtain

Proposition 11 (i) With respect to a transformation of coordinates (1), the components $G_{1(j)i}^{(1)}$ of the global tensor G_1 transform by the rules

$$2\tilde{G}_{1(k)r}^{(1)} = 2G_{1(j)i}^{(1)} \frac{d\tilde{t}}{dt} \frac{\partial x^i}{\partial \tilde{x}^r} \frac{\partial x^j}{\partial \tilde{x}^k} - \frac{\partial x^i}{\partial \tilde{x}^r} \frac{\partial \tilde{p}_k^1}{\partial t} p_i^1. \quad (4)$$

(ii) Conversely, giving a temporal semispray on $J^{1*}(\mathbb{R}, M)$ is equivalent to giving a set of local functions $G_1 = \left(G_{1(j)i}^{(1)} \right)$ which transform by the rules (4).

Example 12 If $H_{11}^1(t) = (h^{11}/2)(dh_{11}/dt)$ is the Christoffel symbol of a semi-Riemannian metric $h_{11}(t)$ of the time manifold \mathbb{R} , then the local components

$$\overset{0}{G}_{1(j)k}^{(1)} = \frac{1}{2} H_{11}^1 p_j^1 p_k^1 \quad (5)$$

represent a temporal semispray $\overset{0}{G}_1$ on the dual 1-jet vector bundle $J^{1*}(\mathbb{R}, M)$.

The temporal semispray $\overset{0}{G}_1$ given by (5) is called the **canonical temporal semispray associated with the metric $h_{11}(t)$** .

A second example of tensor on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, which is not a distinguished tensor, is given by

Definition 13 A global tensor G_2 on $J^{1*}(\mathbb{R}, M)$, locally expressed by

$$G_2 = \delta_i^j dx^i \otimes \frac{\partial}{\partial x^j} - 2G_{(j)i}^{(1)} dx^i \otimes \frac{\partial}{\partial p_j^1},$$

is called a **spatial semispray** on the dual 1-jet vector bundle $J^{1*}(\mathbb{R}, M)$.

Like in the case of a temporal semispray, we can prove without difficulties the following statement:

Proposition 14 Giving a spatial semispray on $J^{1*}(\mathbb{R}, M)$ is equivalent to giving a set of local functions $G_2 = \left(G_{(j)i}^{(1)}\right)$ which transform by the rules

$$2\tilde{G}_{2(s)k}^{(1)} = 2G_{2(j)i}^{(1)} \frac{d\tilde{t}}{dt} \frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial x^j}{\partial \tilde{x}^s} - \frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial \tilde{p}_s^1}{\partial x^i}. \quad (6)$$

Example 15 If $\gamma_{jk}^i(x)$ are the Christoffel symbols of a semi-Riemannian metric $\varphi_{ij}(x)$ of the spatial manifold M , then the local components

$$\overset{0}{G}_{2(j)k}^{(1)} = -\frac{1}{2}\gamma_{jk}^i p_i^1 \quad (7)$$

define a spatial semispray $\overset{0}{G}_2$ on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$. The spatial semispray $\overset{0}{G}_2$ given by (7) is called the **canonical spatial semispray associated with the metric $\varphi_{ij}(x)$** .

Definition 16 A pair $G = \left(G_1, G_2\right)$, consisting of a temporal semispray G_1 and a spatial semispray G_2 , is called a **time-dependent semispray of momenta** on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$.

4 Nonlinear connections and adapted bases

In what follows, we introduce the important geometrical concept of *nonlinear connection* on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$.

Definition 17 A pair of local functions $N = \left(N_{(k)1}^{(1)}, N_{(k)i}^{(1)} \right)$ on $J^{1*}(\mathbb{R}, M)$, which transform by the rules

$$\begin{aligned}\tilde{N}_{(j)1}^{(1)} &= N_{(k)1}^{(1)} \frac{\partial x^k}{\partial \tilde{x}^j} - \frac{dt}{d\tilde{t}} \frac{\partial \tilde{p}_j^1}{\partial t}, \\ \tilde{N}_{(j)r}^{(1)} &= N_{(k)i}^{(1)} \frac{d\tilde{t}}{dt} \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial x^i}{\partial \tilde{x}^r} - \frac{\partial x^i}{\partial \tilde{x}^r} \frac{\partial \tilde{p}_j^1}{\partial x^i},\end{aligned}\tag{8}$$

is called a **nonlinear connection** on the dual 1-jet bundle $J^{1*}(\mathbb{R}, M)$. The geometrical entity $N_1 = \left(N_{(j)1}^{(1)} \right)$ (respectively $N_2 = \left(N_{(j)i}^{(1)} \right)$) is called a **temporal** (respectively **spatial**) **nonlinear connection** on $J^{1*}(\mathbb{R}, M)$.

Now, let us expose the connection between the time-dependent semisprays of momenta and nonlinear connections on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$. For that, let us consider that $\varphi_{ij}(x)$ is a semi-Riemannian metric on the spatial manifold M . Thus, using the transformation rules (4), (6) and (8) of the geometrical objects taken in study, we can easily prove the following statements:

Proposition 18 (i) The temporal semisprays $G_1 = \left(G_{(j)k}^{(1)} \right)$ and the sets of temporal components of nonlinear connections $N_{\text{temporal}} = \left(N_{(r)1}^{(1)} \right)$ are connected on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, via the relations

$$N_{(r)1}^{(1)} = \varphi^{jk} \frac{\partial G_{(j)k}^{(1)}}{\partial p_i^1} \varphi_{ir}, \quad G_{(i)j}^{(1)} = \frac{1}{2} N_{(i)1}^{(1)} p_j^1.$$

(ii) The spatial semisprays $G_2 = \left(G_{(j)i}^{(1)} \right)$ and the sets of spatial components of nonlinear connections $N_{\text{spatial}} = \left(N_{(j)i}^{(1)} \right)$ are in one-to-one correspondence on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, via

$$N_{(j)i}^{(1)} = 2G_{(j)i}^{(1)}, \quad G_{(j)i}^{(1)} = \frac{1}{2} N_{(j)i}^{(1)}.$$

Remark 19 The previous Proposition emphasizes that a time-dependent semispray of momenta $G = \left(G_1, G_2 \right)$ on $J^{1*}(\mathbb{R}, M)$ naturally induces a nonlinear connection N_G on $J^{1*}(\mathbb{R}, M)$ and vice-versa, N induces G_N . The nonlinear connection N_G is called the **canonical nonlinear connection associated with the time-dependent semispray of momenta G and vice-versa**.

Example 20 The canonical nonlinear connection $N = \left(N_{(i)1}^{(1)}, N_{(i)j}^{(1)} \right)$ produced by the canonical time-dependent semispray of momenta $G = \left(G_1, G_2 \right)$,

associated with the pair of semi-Riemannian metrics $(h_{11}(t), \varphi_{ij}(x))$, has the local components

$$\begin{aligned} \overset{0}{N}_{1(i)1}^{(1)} &= H_{11}^1 p_i^1, & \overset{0}{N}_{2(i)j}^{(1)} &= -\gamma_{ij}^k p_k^1. \end{aligned} \quad (9)$$

The nonlinear connection, whose local functions are given by (9), is called the **canonical nonlinear connection on $J^{1*}(\mathbb{R}, M)$, attached to the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$** .

Taking into account the complicated transformation rules (2) and (3), we need a *horizontal distribution* on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, in order to construct some *adapted bases of vector and covector fields*, whose transformation rules are simpler (tensorial ones, for instance).

In this direction, let $u^* = (t, x^i, p_i^1) \in J^{1*}(\mathbb{R}, M)$ be an arbitrary point and let us consider the differential map

$$\pi_{*, u^*}^* : T_{u^*} J^{1*}(\mathbb{R}, M) \rightarrow T_{(t, x)}(\mathbb{R} \times M)$$

of the canonical projection

$$\pi^* : J^{1*}(\mathbb{R}, M) \rightarrow \mathbb{R} \times M, \quad \pi^*(u^*) = (t, x),$$

together with its vector subspace $W_{u^*} = \text{Ker} \pi_{*, u^*}^* \subset T_{u^*} J^{1*}(\mathbb{R}, M)$. Because the differential map π_{*, u^*}^* is a surjection, we find that we have $\dim_{\mathbb{R}} W_{u^*} = n$ and, moreover, a basis in W_{u^*} is determined by $\left\{ \frac{\partial}{\partial p_i^1} \Big|_{u^*} \right\}$.

So, the map $\mathcal{W} : u^* \in J^{1*}(\mathbb{R}, M) \rightarrow W_{u^*} \subset T_{u^*} J^{1*}(\mathbb{R}, M)$ is a differential distribution, which is called the *vertical distribution* on the dual 1-jet vector bundle $J^{1*}(\mathbb{R}, M)$.

Definition 21 A differential distribution

$$\mathcal{H} : u^* \in J^{1*}(\mathbb{R}, M) \rightarrow H_{u^*} \subset T_{u^*} J^{1*}(\mathbb{R}, M),$$

which is supplementary to the vertical distribution \mathcal{W} , that is we have

$$T_{u^*} J^{1*}(\mathbb{R}, M) = H_{u^*} \oplus W_{u^*}, \quad \forall u^* \in J^{1*}(\mathbb{R}, M),$$

is called a **horizontal distribution** on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$.

The above definition implies that $\dim_{\mathbb{R}} H_{u^*} = n + 1, \forall u^* \in J^{1*}(\mathbb{R}, M)$. Moreover, the Lie algebra of the vector fields $\mathcal{X}(J^{1*}(\mathbb{R}, M))$ can be decomposed in the direct sum $\mathcal{X}(J^{1*}(\mathbb{R}, M)) = \mathcal{S}(\mathcal{H}) \oplus \mathcal{S}(\mathcal{W})$, where $\mathcal{S}(\mathcal{H})$ (respectively $\mathcal{S}(\mathcal{W})$) is the set of differentiable sections on \mathcal{H} (respectively \mathcal{W}).

Supposing that \mathcal{H} is a fixed horizontal distribution on $J^{1*}(\mathbb{R}, M)$, we have the isomorphism

$$\pi_{*, u^*}^*|_{H_{u^*}} : H_{u^*} \rightarrow T_{\pi^*(u^*)}(\mathbb{R} \times M),$$

which allows us to prove the following result:

Theorem 22 (i) *There exist unique linear independent horizontal vector fields $\frac{\delta}{\delta t}, \frac{\delta}{\delta x^i} \in \mathcal{S}(\mathcal{H})$, having the properties*

$$\pi^*_* \left(\frac{\delta}{\delta t} \right) = \frac{\partial}{\partial t}, \quad \pi^*_* \left(\frac{\delta}{\delta x^i} \right) = \frac{\partial}{\partial x^i}. \quad (10)$$

(ii) *The horizontal vector fields $\frac{\delta}{\delta t}$ and $\frac{\delta}{\delta x^i}$ can be uniquely written in the form*

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - N_{(j)1}^{(1)} \frac{\partial}{\partial p_j^1}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(j)i}^{(1)} \frac{\partial}{\partial p_j^1}. \quad (11)$$

(iii) *With respect to a transformation of coordinates (1) on $J^{1*}(\mathbb{R}, M)$, the local coefficients $N_{(j)1}^{(1)}$ and $N_{(j)i}^{(1)}$ obey the rules (8) of a nonlinear connection N on $J^{1*}(\mathbb{R}, M)$.*

(iv) *Giving a horizontal distribution \mathcal{H} on $J^{1*}(\mathbb{R}, M)$ is equivalent to giving on $J^{1*}(\mathbb{R}, M)$ a nonlinear connection $N = \left(N_{(j)1}^{(1)}, N_{(j)i}^{(1)} \right)$.*

Proof. Let $\frac{\delta}{\delta t}, \frac{\delta}{\delta x^i} \in \mathcal{X}(J^{1*}(\mathbb{R}, M))$ be vector fields on $J^{1*}(\mathbb{R}, M)$, locally expressed by

$$\begin{aligned} \frac{\delta}{\delta t} &= A_1^1 \frac{\partial}{\partial t} + A_1^j \frac{\partial}{\partial x^j} + A_{(j)1}^{(1)} \frac{\partial}{\partial p_j^1}, \\ \frac{\delta}{\delta x^i} &= X_i^1 \frac{\partial}{\partial t} + X_i^j \frac{\partial}{\partial x^j} + X_{(j)i}^{(1)} \frac{\partial}{\partial p_j^1}, \end{aligned}$$

which verify the relations (10). Then, taking into account the local expression of the map π^*_* , we get

$$\begin{aligned} A_1^1 &= 1, \quad A_1^j = 0, \quad A_{(j)1}^{(1)} = -N_{(j)1}^{(1)}, \\ X_i^1 &= 0, \quad X_i^j = \delta_i^j, \quad X_{(j)i}^{(1)} = -N_{(j)i}^{(1)}. \end{aligned}$$

These equalities prove the form (11) of the vector fields from Theorem, together with their linear independence. The uniqueness of the coefficients $N_{(j)1}^{(1)}$ and $N_{(j)i}^{(1)}$ is obvious.

Because the vector fields $\frac{\delta}{\delta t}$ and $\frac{\delta}{\delta x^i}$ are globally defined, we deduce that a change of coordinates (1) on $J^{1*}(\mathbb{R}, M)$ produces a transformation of the local coefficients $N_{(j)1}^{(1)}$ and $N_{(j)i}^{(1)}$ by the rules (8).

Finally, starting with a set of functions $N = \left(N_{(j)1}^{(1)}, N_{(j)i}^{(1)} \right)$, which respect the rules (8), we can construct the horizontal distribution \mathcal{H} , by putting

$$H_{u^*} = \text{Span} \left\{ \frac{\delta}{\delta t} \Big|_{u^*}, \frac{\delta}{\delta x^i} \Big|_{u^*} \right\}.$$

The decomposition $T_{u^*} J^{1*}(\mathbb{R}, M) = H_{u^*} \oplus W_{u^*}$ is obvious now. ■

Definition 23 *The set of the linear independent vector fields*

$$\left\{ \frac{\delta}{\delta t}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^1} \right\} \subset \mathcal{X}(J^{1*}(\mathbb{R}, M)) \quad (12)$$

*is called the **adapted basis of vector fields produced by the nonlinear connection** $N = \left(N_1, N_2 \right)$.*

With respect to the coordinate transformations (1), the elements of the adapted basis (12) have their transformation laws as tensorial ones (in contrast with the transformations rules (2)):

$$\begin{aligned} \frac{\delta}{\delta t} &= \frac{d\tilde{t}}{dt} \frac{\delta}{\delta \tilde{t}}, \\ \frac{\delta}{\delta x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, \\ \frac{\partial}{\partial p_i^1} &= \frac{d\tilde{t}}{dt} \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{p}_j^1}. \end{aligned}$$

The dual basis (of covector fields) of the adapted basis (12) is given by

$$\{dt, dx^i, \delta p_i^1\} \subset \mathcal{X}^*(J^{1*}(\mathbb{R}, M)) \quad (13)$$

where

$$\delta p_i^1 = dp_i^1 + N_{(i)1}^{(1)} dt + N_{(i)j}^{(1)} dx^j.$$

Definition 24 *The dual basis of covector fields (13) is called the **adapted cobasis of covector fields of the nonlinear connection** $N = \left(N_1, N_2 \right)$.*

Moreover, with respect to transformation laws (1), we obtain the following tensorial transformation rules:

$$\begin{aligned} dt &= \frac{dt}{d\tilde{t}} d\tilde{t}, \\ dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j, \\ \delta p_i^1 &= \frac{dt}{d\tilde{t}} \frac{\partial \tilde{x}^j}{\partial x^i} \delta \tilde{p}_j^1. \end{aligned}$$

As a consequence of the preceding assertions, we find the following simple result:

Proposition 25 (i) *The Lie algebra of vector fields on $J^{1*}(\mathbb{R}, M)$ decomposes in the direct sum $\mathcal{X}(J^{1*}(\mathbb{R}, M)) = \mathcal{X}(\mathcal{H}_{\mathbb{R}}) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{W})$, where*

$$\mathcal{X}(\mathcal{H}_{\mathbb{R}}) = \text{Span} \left\{ \frac{\delta}{\delta t} \right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(\mathcal{W}) = \text{Span} \left\{ \frac{\partial}{\partial p_i^1} \right\}.$$

(ii) *The Lie algebra of covector fields on $J^{1*}(\mathbb{R}, M)$ decomposes in the direct sum $\mathcal{X}^*(J^{1*}(\mathbb{R}, M)) = \mathcal{X}^*(\mathcal{H}_{\mathbb{R}}) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{W})$, where*

$$\mathcal{X}^*(\mathcal{H}_{\mathbb{R}}) = \text{Span} \{dt\}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span} \{dx^i\}, \quad \mathcal{X}^*(\mathcal{W}) = \text{Span} \{\delta p_i^1\}.$$

Definition 26 *The distributions $\mathcal{H}_{\mathbb{R}}$ and \mathcal{H}_M are called the \mathbb{R} -horizontal distribution and M -horizontal distribution on $J^{1*}(\mathbb{R}, M)$.*

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MIRCEA NEAGU
Transilvania University of Brașov
Department of Mathematics and Informatics
Blvd. Iuliu Maniu 50, 500091 Brașov, Romania.
E-mail: mircea.neagu@unitbv.ro